On Product Covering in 3-Tier Supply Chain Models: 

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Abstract

The field of supply chain management has been growing at a rapid pace in recent years, both as a research area and as a practical discipline. In this paper, we study the computational complexity of product covering problems in 3-tier supply chain models, and present natural complete problems for the classes $W[3]$ and $W[4]$ in parameterized complexity theory. This seems the first group of natural complete problems for higher levels in the parameterized intractability hierarchy (i.e., the $W$-hierarchy), and the first precise complexity characterizations of certain optimization problems in the research of supply chain management. Our results also derive strong computational lower bounds and inapproximability for these optimization problems.

1 Introduction

Parameterized complexity theory [8] is a recently proposed and promising approach to the central issue of how to cope with intractable problems - as is so frequently the case in the natural world of computing. An example is the NP-complete problem VERTEX COVER (determining whether a given graph has a vertex cover of size $k$), which now is solvable in time $O(1.285^k + kn)$ [5] and becomes quite practical for various applications. The other direction of the research is the study of parameterized intractability, based on a parameterized intractability hierarchy, the $W$-hierarchy $\bigcup_{t \geq 1} W[t]$. Under a parameterized reduction, the fpt-reduction, a large number of well-known computational problems have been proved to be complete for certain levels of the $W$-hierarchy [8].

For example, CLIQUE, INDEPENDENT SET, SET PACKING, V-C DIMENSION, and WEIGHTED 3-SAT are complete for the class $W[1]$, and DOMINATING SET, HITTING SET, SET COVER, and WEIGHTED SAT are complete for the class $W[2]$. The completeness of a problem in a level of the $W$-hierarchy characterizes precisely the parameterized complexity of the problem.

However, no complete problem is known for any level $W[t]$ for $t > 2$, except the generic problems based on weighted satisfiability on bounded depth circuits and their variations [2, 8]. Therefore, it is interesting to know whether high levels of the $W$-hierarchy, which are defined in terms of formal mathematics, catch the complexity of certain natural computational problems.

In this paper, we present natural complete problems for the classes $W[3]$ and $W[4]$, based on computational problems studied in the areas of supply chain management. The study of supply

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*This research is supported in part by NSF under Grants CCR-0311590 and CCF-0430683.

1We note that a similar situation has occurred in the study of the popular polynomial time hierarchy, for which complete problems for the first level $\Sigma_1^P = \text{NP}$ have been extensively studied while the research on natural complete problems for higher level $\Sigma_t^P$ for $t > 1$ has just started recently [14, 15, 16].
chain management has been growing at a rapid pace in recent years, as a research area and as a practical discipline (see recent survey papers [9, 13]). It has provided extremely rich contexts for the definition of new large-scale optimization problems. Efforts to improve supply chain management have gained the attention of academic researchers, along with the enthusiastic support of government and industry. Therefore, our completeness results in the \( W \)-hierarchy for computational problems in the study of supply chains will also contribute to the understanding of this new computation model. Moreover, based on the recent research on parameterized intractability and inapproximability [4], our results also imply directly inapproximability for these problems.

We give a quick review on the related background.

A parameterized problem consists of instances of the form \((x, k)\), where \(x\) is the problem description and \(k\) is an integer called the parameter. A parameterized problem \(Q\) is fixed parameter tractable if it can be solved by an algorithm of running time \(O(f(k)n^{O(1)})\), where \(f\) is a function independent of \(n = |x|\). Denote by FPT the class of all fixed parameter tractable problems.

A \(\Pi_t\)-circuit of \(n\) input variables \(x_1, \ldots, x_n\) is a \((t + 1)\)-leveled circuit in which (1) the 0-th level is a single output gate that is an AND-gate; (2) each level-\(t\) gate is an input gate labeled by either \(x_i\) (a positive literal) or \(\overline{x}_i\) (a negative literal), \(1 \leq i \leq n\); (3) the outputs of a level-\(j\) gate can only be connected to the inputs of level-\((j - 1)\) gates; and (4) AND-gates and OR-gates are organized into \(t\) alternating levels. A circuit is monotone (resp. antimonotone) if all its input gates are labeled by positive literals (resp. negative literals). A circuit represents naturally a boolean function. A truth assignment \(\alpha\) to the variables of a circuit \(C\) satisfies \(C\) if \(\alpha\) makes the output gate of \(C\) have value 1. The weight of an assignment \(\alpha\) is the number of variables assigned value 1 by \(\alpha\).

The problem weighted satisfiability on \(\Pi_t\)-circuits, briefly WCS[t], consists of instances of the form \((C, k)\), where \(C\) is a \(\Pi_t\)-circuit that is satisfied by an assignment of weight \(k\). The \(W\)-hierarchy, \(\bigcup_{t \geq 1} W[t]\), in parameterized complexity theory is defined based on WCS[t] via a new reduction, the fpt-reduction. We say that a parameterized problem \(Q\) is fpt-reducible to another parameterized problem \(Q'\) if there are two recursive functions \(f\) and \(g\), and an algorithm \(A\) of running time bounded by \(f(k)|x|^{O(1)}\), such that on an input \((x, k)\), the algorithm \(A\) produces a pair \((x', k')\), where \(k' \leq g(k)\), and \((x, k)\) is a yes-instance of \(Q\) if and only if \((x', k')\) is a yes-instance of \(Q'\). It is easy to verify that the fpt-reducibility is transitive [8]. For an integer \(t \geq 2\), a parameterized problem \(Q_1\) is in the class \(W[t]\) if \(Q_1\) is fpt-reducible to the problem WCS[t], a parameterized problem \(Q_2\) is \(W[t]\)-hard if the problem WCS[t] is fpt-reducible to \(Q_2\) (or equivalently, if all problems in \(W[t]\) are fpt-reducible to \(Q_2\)), and a parameterized problem \(Q_3\) is \(W[t]\)-complete if \(Q_3\) is in \(W[t]\) and is \(W[t]\)-hard. In particular, the problem WCS[t] is a generic \(W[t]\)-complete problem for \(t \geq 2\).

We briefly review the related concepts in supply chain management, which has been the subject of a growing body of research literature. The readers are referred to [6, 7, 18] for detailed and systematic discussions, and to [9, 10, 13, 19] for more recent progresses. The underlying structure of a supply chain model is a network consisting of various functional units (such as material suppliers, manufactures, storages, marketing/sales and retailers, and customers) and connections between different units (in the means of both material and information). A supply chain may have numerous tiers in the case of that substructure of manufactures forms a lengthy network itself [13]. Supply chain management involves the management of flows between and among the units in a supply chain to maximize total profitability [9]. The research in supply chain management includes the studies in strategic-, tactical-, and operational-level decisions [9]. In particular, tactical-level decisions, which

\[\text{The corresponding definitions for the class } W[1]\text{ are somehow special and not directly related to our discussion, thus are omitted. The readers are referred to [8] for details.}\]
is the subarea directly related to our current paper, are concerned with medium-range planning efforts, such as production and distribution quantity planning among multiple existing facilities, system-wide inventory policies, and distribution frequency decisions between facilities.

2 Three-tier single product cover and \( W[3] \)-completeness

We follow the supply chain model studied in [17], which is a slight generalization of the model studied in [11]. The model is a 3-tier supply chain that consists of three kinds of units: (material) suppliers, (product) manufacturers, and retailers, such that:

1. A supplier can be linked to a manufacturer, and a manufacturer can be linked to a retailer, standing for transportation/transactions between the units (link capacity is assumed unlimited);
2. A supplier can provide certain materials;
3. A manufacturer can produce a product if all needed materials for the product are provided by suppliers linked to the manufacturer;
4. A retailer has supply of a product if a manufacturer linked to the retailer produces the product.

Such a supply chain can be modeled by a directed graph \( G = (S \cup M \cup R, E) \), where each unit is represented as a vertex in \( G \) and each directed edge in \( E \) represents a link between the corresponding units, here \( S \) is the set of all suppliers, \( M \) is the set of all manufacturers, and \( R \) is the set of all retailers. The objective of optimization studied in the current paper on this model is to maximize the charnel profit [6, 18], that is, to study the strategies that ensure that all retailers have supply of certain products they want to carry. In particular, we say that a product covers all retailers if all retailers have supply of that product.

Now suppose that we want to test the market of a new product at the widest range of customers, using as little experimental resource (i.e., suppliers) as possible and without overloading any supplier. For this, we assign at most one kind of material needed for the new product to each supplier and would like that the product covers all retailers. Obviously, the problem is directly related to the complexity of the product, i.e., the number \( k \) of different kinds of materials needed for the product. Formally, the problem can be formulated as the following parameterized problem:

**3-SCM SINGLE-PRODUCT COVER:**

Let \( G = (S \cup M \cup R, E) \) be a supply chain model, and \( k \) an integer, and suppose that we are going to produce a new product that requires \( k \) different kinds of materials. Is it possible to pick \( k \) suppliers, each for a different kind of material, to produce the new product, such that the product covers all retailers?

Before we prove our main result in this section, we first define the problem \textsc{Weighted Satisfiability on Antimotone }\Pi_3\textsc{-Circuits}, shortly \( \text{wCS}^{-3} \). The problem \( \text{wCS}^{-3} \) is a subproblem of the problem \( \text{wCS}[3] \) that requires that in the input pair \((C, k)\) the \( \Pi_3 \)-circuit \( C \) be antimotone (i.e., all input gates of \( C \) be labeled by negative input literals). It is known that the problem \( \text{wCS}^{-3} \) is also \( W[3] \)-complete [8]. Thus, to prove the \( W[3] \)-completeness for the problem \text{3-SCM SINGLE-PRODUCT COVER}, it suffices to derive fpt-reductions between \( \text{wCS}^{-3} \) and \( 3-\text{SCM SINGLE-PRODUCT COVER}.

**Theorem 2.1** The problem \text{3-SCM SINGLE-PRODUCT COVER} is \( W[3] \)-complete.
PROOF. As explained above, we first present an fpt-reduction from \( \text{wcs}^-[3] \) to 3-SCM SINGLE-PRODUCT COVER. Let \((C, k)\) be an instance of \( \text{wcs}^-[3] \), where \( C \) is an antimonotone \( \Pi_2 \)-circuit. Let \( g_0 \) be the output AND-gate of \( C \) (which is at level 0), \( L_1 \) be the set of OR-gates at level 1 in \( C \) (whose outputs are inputs to \( g_0 \)), \( L_2 \) be the set of AND-gates at level 2 in \( C \) (whose outputs are inputs to gates in \( L_1 \)), and \( L_3 \) be the set of input gates in \( C \) (which are inputs to gates in \( L_2 \) and are labeled by negative input literals).

Construct a 3-tier supply chain model \( G = (S \cup M \cup R, E) \) as follows: (1) each retailer \( \rho_i \) in \( R \) corresponds to an OR-gate \( u_i \) in \( L_1 \); (2) each manufacturer \( \mu_i \) in \( M \) corresponds to an AND-gate \( v_i \) in \( L_2 \); (3) each supplier \( \sigma_i \) in \( S \) corresponds to an input gate \( \tau_i \) in \( L_3 \). The vertices in \( G \) are connected in the following way: (1) there is a link from a manufacturer \( \mu_i \) to a retailer \( \rho_j \) if and only if the corresponding AND-gate \( v_i \) is an input to the corresponding OR-gate \( u_j \); and (2) there is a link from a supplier \( \sigma_i \) to a manufacturer \( \mu_j \) if and only if the corresponding input gate \( \tau_i \) is not an input to the corresponding AND-gate \( v_j \) (note that \( C \) is an antimonotone circuit). This completes the description of the 3-tier supply chain model \( G \). We prove that the circuit \( C \) has a satisfying assignment \( \alpha \) of weight \( k \) if and only if we can pick \( k \) suppliers in the supply chain \( G \), each for a different kind of material for a new product that needs \( k \) kinds of materials, so that the new product covers all retailers.

Suppose that the circuit \( C \) has a satisfying assignment \( \alpha \) of weight \( k \). Let \( X_k \) be the set of \( k \) variables in \( C \) that are assigned value 1 by \( \alpha \). Let \( S_k \) be the \( k \) suppliers corresponding to the \( k \) input variables in \( X_k \). We show that we can pick the \( k \) suppliers in \( S_k \), each for a different kind of material for the new product that needs \( k \) kinds of materials, such that the product covers all retailers in \( G \). Consider any manufacturer \( \mu_i \) in \( M \). If \( \mu_i \) has the supply for all \( k \) kinds of materials for the new product, i.e., if \( \mu_i \) has links from all the \( k \) suppliers in \( S_k \), then by the construction of the supply chain model \( G \), the corresponding AND-gate \( v_i \) in \( C \) has no input from any input gate \( \tau_j \) where \( x_j \) is an input variable in \( X_k \). Therefore, under the assignment \( \alpha \), all inputs to the gate \( v_i \) have value 1 and the output of \( v_i \) has value 1. On the other hand, if the manufacturer \( \mu_i \) does not receive supply from a supplier \( \sigma_j \) in \( S_k \), then the input gate \( \tau_j \) is an input to the AND-gate \( v_i \), and under the assignment \( \alpha \), the output of gate \( v_i \) has value 0. In summary, the AND-gate \( v_i \) outputs value 1 if and only if the corresponding manufacturer \( \mu_i \) has supply from all \( k \) suppliers in \( S_k \) and is able to produce the new product. Now, a retailer \( \rho_k \) has supply of the new product if and only if it has a link from a manufacturer \( \mu_j \) that can produce the new product, which by the above analysis if and only if the corresponding AND-gate \( v_j \) in \( L_2 \) outputs value 1 under the assignment \( \alpha \). Since the retailer \( \rho_k \) has a link from a manufacturer \( \mu_j \) if and only if the corresponding OR-gate \( u_i \) in \( C \) has input from the corresponding AND-gate \( v_j \), we conclude that the retailer \( \rho_k \) has supply of the new product if and only if the corresponding OR-gate \( u_i \) in \( C \) outputs value 1 under the assignment \( \alpha \). Finally, since the output AND-gate \( g_0 \) of \( C \) is connected to all OR-gates in \( L_1 \), we conclude that the circuit \( C \) has value 1 if and only if all retailers have supply of the new product. In consequence, if \( \alpha \) is a satisfying assignment for the circuit \( C \), then picking the \( k \) suppliers in \( S_k \) results in the new product that covers all retailers in \( G \).

Conversely, suppose there is a set \( S_k \) of \( k \) suppliers, each for a different kind of material for the new product such that the new product covers all retailers. We let \( X_k \) be the \( k \) input variables in the circuit \( C \) corresponding to the \( k \) suppliers in \( S_k \). Let \( \alpha \) be a weight-\( k \) assignment to \( C \) that assigns value 1 to the \( k \) variables in \( X_k \) and value 0 to all other input variables. Then following exactly the same reasoning as above, we can verify that the assignment \( \alpha \) satisfies the circuit \( C \).
This completes the analysis of the reduction from \(\text{WCS}^{-}[3]\) to 3-SCM SINGLE-PRODUCT COVER. The reduction is obviously an fpt-reduction. In conclusion, we have proved that the problem 3-SCM SINGLE-PRODUCT COVER is \(W[3]\)-hard.

To show that the problem 3-SCM SINGLE-PRODUCT COVER is in \(W[3]\), it suffices to show that 3-SCM SINGLE-PRODUCT COVER is fpt-reducible to \(\text{WCS}^{-}[3]\). The construction is very similar to the one described above: for an instance \((G,k)\) of 3-SCM SINGLE-PRODUCT COVER, where \(G = (S \cup M \cup R, E)\) is a supply chain model and \(k\) is an integer, we construct an instance \((C,k)\) of \(\text{WCS}^{-}[3]\), where each level-1 OR-gate in \(C\) corresponds to a retailer in \(R\), each level-2 AND-gate in \(C\) corresponds to a manufacturer in \(M\), and each input gate in \(C\) (labeled by a negative literal) corresponds to a supplier in \(S\). A level-1 OR-gate has an input from a level-2 AND-gate if and only if the corresponding retailer has a link from the corresponding manufacturer in \(G\), and a level-2 AND-gate has an input from an input gate if and only if the corresponding manufacturer has no link from the corresponding supplier. Now by the exact method, we can verify that the circuit \(C\) has a satisfying assignment of weight \(k\) if and only if there are \(k\) suppliers, each for a different kind of material for the new product, such that the new product covers all retailers. In consequence, the problem 3-SCM SINGLE-PRODUCT COVER is in the class \(W[3]\).

This proves that the problem 3-SCM SINGLE-PRODUCT COVER is \(W[3]\)-complete. \(\square\)

3 Three-tier multiple product cover and \(W[4]\)-completeness

To describe \(W[4]\)-complete problems, we consider a more general model of 3-tier supply chains by allowing a supplier to provide multiple kinds of materials, a manufacturer to produce multiple kinds of products, and a retailer to carry multiple kinds of products.

We first consider a problem that is concerned with the covering by a line of homogenous (i.e., similar) products. Formally, let \(P\) be a given line of homogenous products and let \(T\) be a set of materials, where each product \(\pi\) in \(P\) is associated with a set of materials in \(T\) that are needed for producing the product. In a 3-tier supply chain \(G = (S \cup M \cup R, E)\), each supplier \(\sigma\) in \(S\) is associated with a list of materials in \(T\) that the supplier \(\sigma\) can provide, each manufacturer \(\mu\) in \(M\) is associated with a list of products in \(P\) that the manufacturer \(\mu\) can produce when necessary materials are provided by suppliers linked to \(\mu\), and each retailer \(\rho\) in \(R\) is associated with a suggested list of products in \(P\) that the retailer \(\rho\) is interested in carrying when the products are produced by the manufacturers linked to \(\rho\). We are interested in the following problem in supply chain management: for a new line \(P\) of homogenous products, we want to use limited amount of resource (i.e., a small number of suppliers) to test the product market in the widest range of customers (i.e., make all retailers have supply of some of the new products). This is formulated as the following parameterized problem.

**GENERAL 3-SCM H-PRODUCT-LINE COVER:**

Given a line \(P\) of homogenous products, a general supply chain model \(G = (S \cup M \cup R, E)\), and an integer \(k\), is it possible to pick \(k\) suppliers for the products in \(P\) so that each retailer has supply of some products in its associated product list?

To study the complexity of this problem, we consider the problem \textsc{Weighted Satisfiability on Monotone \(\Pi_1^p\)-circuits}, shortly \(\text{WCS}^+[4]\), which is a subproblem of the problem \(\text{WCS}[4]\) with an additional constraint that in the input pair \((C,k)\) the \(\Pi_1^p\)-circuit \(C\) be monotone (i.e., all input
gates of $C$ be labeled by positive input literals). It is known that the problem $\text{WCS}^+[4]$ is also $W[4]$-complete [8]. Thus, in order to prove the $W[4]$-completeness for GENERAL 3-scm H-PRODUCT-LINE COVER, it suffices to present fpt-reductions between $\text{WCS}^+[4]$ and GENERAL 3-scm H-PRODUCT-LINE COVER.

**Lemma 3.1** The problem GENERAL 3-scm H-PRODUCT-LINE COVER is in $W[4]$.

**Proof.** We show how the problem GENERAL 3-scm H-PRODUCT-LINE COVER is fpt-reducible to the problem $\text{WCS}^+[4]$.

Let $(P, G, k)$ be an instance of GENERAL 3-scm H-PRODUCT-LINE COVER, where $P = \{\pi_1, \ldots, \pi_h\}$ is a product line and each product $\pi_i$ is associated with a set of materials needed for producing $\pi_i$, $G = (S \cup M \cup R, E)$ is a general 3-tier supply chain with the supplier set $S = \{\sigma_1, \ldots, \sigma_n\}$, the manufacturer set $M = \{\mu_1, \ldots, \mu_m\}$, and the retailer set $R = \{\rho_1, \ldots, \rho_t\}$. Let $T = \{\tau_1, \ldots, \tau_p\}$ be the set of different materials that are needed for the products in $P$ ($T$ can be obtained directly from the product list $P$). Each supplier $\sigma_i$ is associated with a subset of $T$, indicating the materials that can be provided by $\sigma_i$; each manufacturer $\mu_i$ is associated with a subset of $P$, indicating the products that can be produced by $\mu_i$ when necessary materials are provided by the suppliers linked to $\mu_i$; and each retailer $\rho_i$ is associated with a subset of $P$, indicating the products that the retailer $\rho_i$ is interested in carrying from the manufacturers linked to $\rho_i$.

We construct an instance $(C, k)$ for $\text{WCS}^+[4]$, where the monotone $\Pi_4$-circuit $C$ has the following structure:

(V0) The level-0 output AND-gate in $C$ is $g_0$;

(V1) The set of level-1 gates in $C$ consists of $t$ OR-gates $u_i$, $1 \leq i \leq t$, corresponding to the $t$ retailers in $G$;

(V2) For each manufacturer $\mu_i$ and each product $\pi_j$ in the associated product list of $\mu_i$, there is an AND-gate $v_{ij}$ in level 2 in $C$;

(V3) For each manufacturer $\mu_i$ and each material $\tau_l$ that is needed for a product in the associated product list of $\mu_i$, there is an OR-gate $w_{ij}$ in level 3 in $C$;

(V4) the set of level-4 gates in $C$ consists of $n$ input gates labeled $x_i$, $1 \leq i \leq n$, respectively, corresponding to the $n$ suppliers in $G$.

The gates in the circuit $C$ are connected as follows:

(E1) all level-1 gates are inputs to the output gate $g_0$;

(E2) a level-2 gate $v_{ij}$ is an input to a level-1 gate $u_s$ if there is a link in $G$ from the manufacturer $\mu_i$ to the retailer $\rho_s$ in the supply chain $G$, and if the product $\pi_j$ is contained in both associated product lists of $\mu_i$ and $\rho_s$;

(E3) a level-3 gate $w_{ij}$ is an input to a level-2 gate $v_{is}$ if the material $\tau_j$ is needed for the product $\pi_s$, and the product $\pi_j$ is contained in the associated product list of the manufacturer $\mu_i$;

(E4) an input gate labeled $x_i$ is an input of a level-3 gate $w_{js}$ if and only if the supplier $\sigma_i$ can provide the material $\tau_s$ and if the supplier $\sigma_i$ is linked to the manufacturer $\mu_j$ (note that by the construction, the material $\tau_s$ is needed for some product in $\mu_j$).

This completes the description of the circuit $C$, which is obviously a monotone $\Pi_4$-circuit. We now show that $(P, G, k)$ is a yes-instance for GENERAL 3-scm H-PRODUCT-LINE COVER if and only if $(C, k)$ is a yes-instance for $\text{WCS}^+[4]$. For this, we establish a one-to-one mapping between the subsets of $k$ suppliers in $G$ and the weight-$k$ assignments to the circuit $C$, as follows: each subset $S_k$ of $k$ suppliers in $G$ corresponds to the assignment $\phi(S_k)$ in $C$ that assigns value 1 to an input variable $x_i$ if and only if the supplier $\sigma_i$ is in $S_k$. To prove the lemma, it suffices to show that $\phi(S_k)$
is a satisfying assignment for $C$ if and only if picking the $k$ suppliers in $S_k$ will make all retailers in $G$ have supply for some products in $P$. This can be verified by the following facts:

(F1) By rule (E4), a level-3 OR-gate $w_{js}$ has value 1 under the assignment $\phi(S_k)$ if and only if the manufacturer $\mu_j$ has supply of material $\tau_s$ under the supplier selection $S_k$;

(F2) By rule (E3), a level-2 AND-gate $v_{is}$ has value 1 under the assignment $\phi(S_k)$ if and only if the manufacturer $\mu_i$ has supply for all needed materials for product $\pi_s$ under the supplier selection $S_k$, that is, if and only if $\mu_i$ can produce the product $\pi_s$;

(F3) By rule (E2), a level-1 OR-gate $u_s$ has value 1 under the assignment $\phi(S_k)$ if and only if the retailer $\rho_s$ has supply of some products in its associated product list under the supplier selection $S_k$.

Summarizing facts (F1), (F2), and (F3), we conclude that the weight-$k$ assignment $\phi(S_k)$ satisfies the circuit $C$ if and only if all retailers in $G$ have supply for some products in $P$ under the selection of the $k$ suppliers in $S_k$. Since the mapping from $S_k$ to $\phi(S_k)$ is a one-to-one mapping from the subsets of $k$ suppliers in $G$ to the weight-$k$ assignments for the circuit $C$, this verifies that this reduction from GENERAL 3-scm H-PRODUCT-LINE COVER to WCs$^+[4]$ is an fpt-reduction. In consequence, the problem GENERAL 3-scm H-PRODUCT-LINE COVER is in the class W[4].

Now we verify the W[4]-hardness for the problem GENERAL 3-scm H-PRODUCT-LINE COVER.

**Lemma 3.2** The problem GENERAL 3-scm H-PRODUCT-LINE COVER is W[4]-hard.

**Proof.** As explained in the paragraph before Lemma 3.1, it suffices to present an fpt-reduction from the problem WCs$^+[4]$ to the problem GENERAL 3-scm H-PRODUCT-LINE COVER.

Let $(C, k)$ be an instance of WCs$^+[4]$, where $C$ is a monotone $\Pi_1$-circuit. Let the output AND-gate of $C$ be $g_0$, and suppose that in the circuit $C$ there are $t$ level-1 OR-gates $u_i$, $1 \leq i \leq t$, $q$ level-2 AND-gates $v_i$, $1 \leq i \leq q$, $m$ level-3 OR-gates $w_i$, $1 \leq i \leq m$, and $n$ input gates labeled $x_i$, $1 \leq i \leq n$. We first perform a preprocessing on the circuit $C$ as follows. If any gate $g$ (at any level) has exactly the same input as another gate $g'$ at the same level, then we “merge” these two gates into a single gate $g''$ of the same type, and let the output of $g''$ connect to the outputs of $g$ and $g'$ in the original circuit. It is easy to see that such a modification can be done in polynomial time and does not change the circuit function. With this preprocessing, we can assume, without loss of generality, that no two gates at the same level in the circuit have exactly the same input.

We construct an instance $(P, G, k)$ for GENERAL 3-scm H-PRODUCT-LINE COVER, where $G = (S \cup M \cup R, E)$, as follows. Each level-3 gate $w_i$ corresponds to a different material $\tau_i$, $1 \leq i \leq m$, and each input variable $x_i$ corresponds to a supplier $\sigma_i$, $1 \leq i \leq n$ (thus, $S = \{\sigma_1, \ldots, \sigma_n\}$). A material $\tau_i$ is in the supplier $\sigma_j$ (recall that each supplier is specified by a set of materials that can be provided by the supplier) if and only if the variable $x_j$ is an input to the level-3 gate $w_i$. There are $q$ products $\pi_i$ in $P$, $1 \leq i \leq q$, such that a material $\tau_j$ is needed for the product $\pi_i$ if and only if the level-3 gate $w_j$ is an input to the level-2 gate $v_i$ (thus, $P = \{\pi_1, \ldots, \pi_q\}$, note that by our preprocessing, all products in $P$ require different subsets of materials). There are also $q$ manufacturers $\mu_i$, corresponding to the $q$ level-2 gates $v_i$ in $C$, $1 \leq i \leq q$ (thus, $M = \{\mu_1, \ldots, \mu_q\}$). For each $i$, the manufacturer $\mu_i$ is associated with the product set $\{\pi_i\}$ consisting of a single product $\pi_i$. Finally, there are $t$ retailers $\rho_i$, corresponding to the $t$ level-1 gates $u_i$ in $C$, $1 \leq i \leq t$ (thus, $R = \{\rho_1, \ldots, \rho_t\}$). The product set associated with a retailer $\rho_i$ consists of exactly those products $\pi_j$ such that the level-2 gate $v_j$ is an input of the level-1 gate $u_i$. The suppliers in $S$ and the manufacturers in $M$ are fully connected (i.e., every supplier is linked to every manufacturer), and
a manufacturer $\mu_i$ is linked to a retailer $\rho_j$ if and only if the level-2 gate $v_i$ is an input to the level-1 gate $u_j$. This completes the description of the general 3-tier supply chain model $G$ and the product set $P$.

As we did in Lemma 3.1, we establish a one-to-one mapping between the subsets of $k$ suppliers in $G$ and the weight-$k$ assignments for the circuit $C$, by mapping a subset $S_k$ of $k$ suppliers to the assignment $\phi(S_k)$ such that $\phi(S_k)$ assigns value 1 to a variable $x_i$ if and only if the supplier $\sigma_i$ is in the subset $S_k$. Again we show that $\phi(S_k)$ is a satisfying assignment for the circuit $C$ if and only if the selection of the $k$ suppliers in $S_k$ makes all retailers in $R$ to have supply for some products in $P$. This can be proved by verifying the following facts.

(F1) By our construction of the suppliers in $S$, and since the suppliers and the manufacturers in $G$ are fully connected, a level-3 OR-gate $w_i$ in $C$ has value 1 under the assignment $\phi(S_k)$ if and only if the material $\tau_i$ can be provided (to all manufacturers) under the supplier selection $S_k$;

(F2) A level-2 AND-gate $v_i$ has value 1 under the assignment $\phi(S_k)$ if all of its inputs in level 3 have value 1, equivalently, if all needed materials for product $\tau_i$ are provided for the manufacturer $\mu_i$ under the supplier selection $S_k$. Therefore, a level-2 gate $v_i$ has value 1 under the assignment $\phi(S_k)$ if and only if the manufacturer $\mu_i$ can produce the product $\tau_i$ under the supplier selection $S_k$ (note that $\tau_i$ is the only product that can be produced by the manufacturer $\mu_i$);

(F3) Finally, by our connections between the manufacturers and retailers in $G$, a retailer $\rho_i$ has supply of a product $\tau_j$ in $P$ under the supplier selection $S_k$ if and only if the manufacturer $\mu_j$ is linked to $\rho_i$ and can produce the product $\tau_j$, equivalently, if and only if the level-2 gate $v_j$ in $C$ is an input of the level-1 OR-gate $u_i$ and $v_j$ has value 1 under the assignment $\phi(S_k)$. Therefore, the retailer $\rho_i$ has supply of some products in $P$ under the supplier selection $S_k$ if and only if the level-1 gate $u_i$ has value 1 under the assignment $\phi(S_k)$.

This completes the verification that the assignment $\phi(S_k)$ satisfies the circuit $C$ if and only if every retailer in $G$ has supply of some products in $P$. It clearly gives an fpt-reduction from WCST$^+$[4] to GENERAL 3-scm H-PRODUCT-LINE COVER. In consequence, the problem GENERAL 3-scm H-PRODUCT-LINE COVER is W[4]-hard.

Combining Lemma 3.1 and Lemma 3.2, we get immediately,

**Theorem 3.3** The problem GENERAL 3-scm H-PRODUCT-LINE COVER is W[4]-complete.

The W[4]-completeness also provides precise complexity characterization for other computational problems in 3-tier supply chain management. For example, suppose now that a firm is interested in investigating the market for a set $P$ of non-homogenous products. The 3-tier supply chain is again given as a network of suppliers, manufacturers, and retailers, where each supplier is given as before and associated with a set of materials that can be provided by the supplier. Each manufacturer $\mu$ is associated with a set $T_\mu$ of materials and a set $P_\mu$ of products such that when all materials in $T_\mu$ are provided by suppliers linked to $\mu$, the manufacturer $\mu$ can produce all products in $P_\mu$. Finally, each retailer $\rho$ is associated with a requested list of products that must be carried by the retailer $\rho$. This supply chain model gives a parameterized problem as follows:

**GENERAL 3-scm PRODUCT-SET COVER:**

Given a product set $P$, a general supply chain model $G = (S \cup M \cup R, E)$ as described above, and an integer $k$, is it possible to pick $k$ suppliers in $S$ for materials so that every retailer in $R$ has supply of all products in its associated product list?
The main difference between GENERAL 3-SCM H-PRODUCT-LINE COVER and GENERAL 3-SCM PRODUCT-SET COVER is that in the former model each retailer only needs to carry some of the products in its associated list while in the latter model each retailer must carry all products in its associated list.

The proof for the following theorem is similar to (actually, slightly simpler than) that for Lemma 3.1 and Lemma 3.2. We omit it and leave it to the interested readers.

**Theorem 3.4** The GENERAL 3-SCM PRODUCT-SET COVER problem is $W[4]$-complete.

4 Computational lower bounds and inapproximability results

Theorem 2.1, Theorem 3.3 and Theorem 4.1 provide strong lower bounds for the complexity of the problems 3-SCM SINGLE-PRODUCT COVER, GENERAL 3-SCM H-PRODUCT-LINE COVER, and GENERAL 3-SCM PRODUCT-SET COVER.

**Theorem 4.1** For any recursive function $f$, the problem 3-SCM SINGLE-PRODUCT COVER cannot be solved in time $f(k)m^{O(1)}n^{o(k)}$ unless $W[2] = FPT$, and the problems GENERAL 3-SCM H-PRODUCT-LINE COVER and GENERAL 3-SCM PRODUCT-SET COVER cannot be solved in time $f(k)m^{O(1)}n^{o(k)}$ unless $W[3] = FPT$, where $n$ is the number of suppliers and $m$ is the size of the instance of the problems.

**Proof.** Suppose that the problem 3-SCM SINGLE-PRODUCT COVER could be solved in time $f(k)m^{O(1)}n^{o(k)}$, then by the fpt-reduction from $WCS^{\neg}[3]$ to 3-SCM SINGLE-PRODUCT COVER given in Theorem 2.1, it is easy to see that the problem $WCS^{\neg}[3]$ can also be solved in time $f(k)m^{O(1)}n^{o(k)}$, where $m$ is the instance size and $n$ is the number of input variables in the circuit. By Theorem 4.2 in [4], it would imply $W[2] = FPT$. The lower bounds for GENERAL 3-SCM H-PRODUCT-LINE COVER and GENERAL 3-SCM PRODUCT-SET COVER can be proved in the same way using the same theorem in [4].

Since it is generally believed that $W[t] \neq FPT$ for all $t > 0$, Theorem 4.1 provides a computational lower bound $f(k)m^{O(1)}n^{\Omega(k)}$ for the problems 3-SCM SINGLE-PRODUCT COVER, GENERAL 3-SCM H-PRODUCT-LINE COVER, and GENERAL 3-SCM PRODUCT-SET COVER. Note that this is an asymptotically tight lower bound for the problems as the algorithm that exhaustively enumerates and examines all subsets of $k$ suppliers in a problem instance solves the problems in time $O(m^2n^k)$ trivially.

Theorem 4.1 further implies inapproximability results for certain optimization problems in 3-tier supply chain management. For this, we need to first review some related terminologies in approximation algorithms. The readers are referred to [1] for more detailed definitions and more comprehensive discussions.

An optimization problem $Q$ consists of a set of instances, where each instance $x$ is associated with a set of solutions. Each solution $y$ of an instance $x$ of $Q$ is assigned an integral value $f_Q(x,y)$. The problem $Q$ is a maximization (resp. minimization) problem if for each instance $x$ of $Q$, we are looking for a solution of maximum (resp. minimum) value. Such a solution is called an optimal solution for the instance, whose value is denoted by $opt_Q(x)$.

An algorithm $A$ is an approximation algorithm for an optimization problem $Q$ if, for each instance $x$ of $Q$, the algorithm $A$ returns a solution $y_A(x)$ for $x$. The solution $y_A(x)$ has an
approximation ratio $r$ if it satisfies the following condition:

$$\frac{\text{opt}_Q(x)}{f_Q(x, y_A(x))} \leq r \text{ if } Q \text{ is a maximization problem}$$

$$\frac{f_Q(x, y_A(x))}{\text{opt}_Q(x)} \leq r \text{ if } Q \text{ is a minimization problem}$$

The approximation algorithm $A$ has an approximation ratio $r$ if for any instance $x$ of $Q$, the solution $y_A(x)$ constructed by the algorithm $A$ has an approximation ratio bounded by $r$. A polynomial time approximation scheme (PTAS) for $Q$ is an algorithm $A'$ that on an instance $x$ of $Q$ and a real number $\epsilon > 0$, constructs a solution for $x$ whose approximation ratio is bounded by $1 + \epsilon$, and the running time of $A'$ is bounded by a polynomial of $|x|$ for each fixed $\epsilon$ [1].

Now consider the following optimization problems in supply chain management:

**3-scm most-complicated product cover**

Given a 3-tier supply chain $G$, select the largest number $k$ of suppliers in $G$, each for a different kind of material, such that a new product that needs the $k$ materials can be produced and all retailers in $G$ have supply of the new product.

**General 3-scm minimum-resource H-product-line cover**

Given a line $P$ of homogenous products and a general 3-tier supply chain $G$ (as defined in General 3-scm H-product-line cover), select the minimum number of suppliers in $G$ for the product line $P$, such that each retailer in $G$ has supply of some products in its associated product list.

**General 3-scm minimum-resource product-set cover**

Given a set $P$ of non-homogenous products and a general 3-tier supply chain $G$ (as defined in General 3-scm product-set cover), select the minimum number of suppliers in $G$ for the product set $P$, such that each retailer in $G$ has supply of all products in its product list.

Note that 3-scm most-complicated product cover is a maximization problem while General 3-scm minimum-resource H-product-line cover and General 3-scm minimum-resource product-set cover are minimization problems.

An optimization problem $Q$ can be parameterized using the following formulation [4]:

**Definition** The parameterized version of an optimization problem $Q$ is defined as follows:

1. If $Q$ is a maximization problem, then the parameterized version of $Q$ is defined as $Q_\geq = \{(x, k) \mid x \text{ is an instance of } Q \text{ and } \text{opt}_Q(x) \geq k\}$.
2. If $Q$ is a minimization problem, then the parameterized version of $Q$ is defined as $Q_\leq = \{(x, k) \mid x \text{ is an instance of } Q \text{ and } \text{opt}_Q(x) \leq k\}$.

In particular, it is easy to see that the parameterized version of the problem 3-scm most-complicated product cover is equivalent to the problem 3-scm single-product cover, the parameterized version of the problem general 3-scm minimum-resource H-product-line cover is equivalent to the problem general 3-scm H-product-line cover, and the parameterized version of the problem general 3-scm minimum-resource product-set cover is equivalent to the problem general 3-scm product-set cover. Now we are ready for the following theorem.
Theorem 4.2 For any recursive function $f$, the problem $3$-SCM MOST-COMPPLICATED PRODUCT COVER has no PTAS of running time $f(1/e)m^{O(1)}n^{O(1/e)}$ unless $W[2] = FPT$, and the problems $3$-SCM GENERAL 3-SCM MINIMUM-RESOURCE H-PRODUCT-LINE COVER and $3$-SCM GENERAL 3-SCM MINIMUM-RESOURCE PRODUCT-SET COVER have no PTAS of running time $f(1/e)m^{O(1)}n^{O(1/e)}$ unless $W[3] = FPT$, where $n$ is the number of suppliers and $m$ is the instance size of the problems.

Proof. Suppose that the problem $3$-SCM MOST-COMPPLICATED PRODUCT COVER has a PTAS of running time $f(1/e)m^{O(1)}n^{O(1/e)}$, then by Theorem 5.1 in [3], its parameterized version, i.e., the problem $3$-SCM SINGLE-PRODUCT COVER can be solved in time $f(2k)m^{O(1)}n^{O(k)}$, which, by Theorem 4.1, would imply $W[2] = FPT$. The inapproximability for $3$-SCM GENERAL 3-SCM MINIMUM-RESOURCE H-PRODUCT-LINE COVER and $3$-SCM GENERAL 3-SCM MINIMUM-RESOURCE PRODUCT-SET COVER can be proved using the same logic.

Since it is commonly believed in parameterized complexity theory that $W[t] \neq FPT$ for all $t \geq 1$, Theorem 4.2 implies that even for a moderate error bound $\epsilon > 0$, any PTAS for the problems, if exists, will become impractical.

5 Final remarks

This paper studies the complexity issues for certain computational problems arising from the research in supply chain management, and characterizes these problems in terms of parameterized completeness in higher levels in the $W$-hierarchy. The research contributes to both parameterized complexity theory and the study of supply chain management. For parameterized complexity theory, we presented the first group of natural complete problems for the classes $W[3]$ and $W[4]$, which had no known natural complete problems except the generic complete problems $WCS[3]$ and $WCS[4]$ and their variations. For the study of supply chain management, to authors' knowledge, our results provide first group of precise complexity characterizations for certain computational problems in the area, which derive directly strong computational lower bounds and inapproximability results for the problems. The hardness results of these problems will provide useful information in the study of supply chain management.

A supply chain model has its units classified into different kinds, which makes it natural to map the computation in the supply chain model to that of bounded depth circuits. However, the mapping is not always straightforward and in many cases must be with care. As we have seen in the current paper, problems on 3-tier supply chains can either correspond to the class $W[3]$, which is associated with the satisfiability problem on $\Pi_3$-circuits of 3 levels, or correspond to the class $W[4]$, which is associated with the satisfiability problem on $\Pi_4$-circuits of 4 levels. Our more recent research studied a computational problem, HARMFUL WASTE MAKERS, on the recycling system proposed in [12]. The problem is concerned with whether there are $k$ waste makers who can pollute all markets. This recycling system is a 4-tier supply chain model, consisting of waste makers, recycling centers, processors, and markets. However, our study shows that the problem HARMFUL WASTE MAKERS is actually $W[2]$-complete (i.e., corresponding to the satisfiability problem on $\Pi_2$-circuits of 2 levels). In conclusion, the computational complexity of the problems in supply chain management does not directly depend on the number of tiers in the model but is more closely related to the actual applications. In particular, the research in supply chain management has opened an area in computational complexity and optimization, and provided very rich contexts
for new large-scale optimization problems that are both theoretically interesting and practically important.

References


